Convetions

Έστω $\mathfrak{a}, \mathfrak{b}$ γραμμικοί υπόχωροι της Lie άλγεβρας $\mathfrak{g}$.

$$[\mathfrak{a},\mathfrak{b}] \overset{def}{=} span_{\mathbb{C}}\{[a,b] \text{ τ.ω. } \forall a\in\mathfrak{a}, \forall b\in\mathfrak{b}\}$$ δχδ

Τα στοιχεία του $[\mathfrak{a},\mathfrak{b}]$ είναι της μορφής

$$\sum_i [a_i,b_i] \text{ όπου } a_i\in\mathfrak{a},\ b_i\in\mathfrak{b}.$$

- Το $[\mathfrak{a},\mathfrak{b}]$ είναι γραμμικός χώρος
- $[\mathfrak{a},\mathfrak{b}] = [\mathfrak{b},\mathfrak{a}]$

Subalgebras

Definition (Lie subalgebra)

$\mathfrak{h}$ **Lie subalgebra** $\mathfrak{g} \Leftrightarrow$ (i) $\mathfrak{h}$ vector subspace of $\mathfrak{g}$
(ii) $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$

Examples of $\mathfrak{gl}(n, \mathbb{C})$ subalgebras

Diagonal matrices $\mathfrak{d}(n, \mathbb{C})$= matrices with diagonal elements only

$$[\mathfrak{d}(n, \mathbb{C}), \mathfrak{d}(n, \mathbb{C})] = 0_n \subset \mathfrak{d}(n, \mathbb{C})$$

Upper triangular matrices $\mathfrak{t}(n, \mathbb{C})$,
Strictly Upper triangular matrices $\mathfrak{n}(n, \mathbb{C})$,

$$\mathfrak{n}(n, \mathbb{C}) \subset \mathfrak{t}(n, \mathbb{C})$$

$$[\mathfrak{t}(n, \mathbb{C}), \mathfrak{t}(n, \mathbb{C})] \subset \mathfrak{n}(n, \mathbb{C}), \quad [\mathfrak{n}(n, \mathbb{C}), \mathfrak{n}(n, \mathbb{C})] \subset \mathfrak{n}(n, \mathbb{C})$$

$$[\mathfrak{n}(n, \mathbb{C}), \mathfrak{t}(n, \mathbb{C})] \subset \mathfrak{n}(n, \mathbb{C})$$

ex. $\mathfrak{sl}(2, \mathbb{C})$ Lie subalgebra of $\mathfrak{sl}(3, \mathbb{C})$

$\mathfrak{sl}(3, \mathbb{C})$

Algebra $\mathfrak{sl}(3,\mathbb{C}) = \mathrm{span}\,(h_1,\ h_2,\ x_1,\ y_1,\ x_2,\ y_2,\ x_3,\ y_3)$

$$
\begin{array}{lll}
[h_1,\ h_2] = 0 & & \\
[h_1,\ x_1] = 2x_1, & [h_1,\ y_1] = -2y_1, & [x_1,\ y_1] = h_1, \\
[h_1,\ x_2] = -x_2, & [h_1,\ y_2] = y_2, & \\
[h_1,\ x_3] = x_3, & [h_1,\ y_3] = -y_3, & \\
[h_2,\ x_2] = 2x_2, & [h_2,\ y_2] = -2y_2, & [x_2,\ y_2] = h_2, \\
[h_2,\ x_1] = -x_1, & [h_2,\ y_1] = y_1, & \\
[h_2,\ x_3] = x_3, & [h_2,\ y_3] = -y_3, & \\
[x_1,\ x_2] = x_3, & [x_1,\ x_3] = 0, & [x_2,\ x_3] = 0 \\
[y_1,\ y_2] = -y_3, & [y_1,\ y_3] = 0, & [y_2,\ y_3] = 0 \\
[x_1,\ y_2] = 0, & [x_1,\ y_3] = -y_2, & \\
[x_2,\ y_1] = 0, & [x_2,\ y_3] = y_1, & \\
[x_3,\ y_1] = -x_2, & [x_3,\ y_2] = x_1, & [x_3,\ y_3] = h_1 + h_2
\end{array}
$$

$\Rightarrow \mathrm{span}_{\mathbb{C}}\{h_1, x_1, y_1\} \overset{\text{iso}}{=} \mathfrak{sl}(2,\mathbb{C})$

$\Rightarrow \mathrm{span}_{\mathbb{C}}\{h_2, x_2, y_2\} \overset{\text{iso}}{=} \mathfrak{sl}(2,\mathbb{C})$

Αν $h_3 = h_1 + h_2$

$\Rightarrow \mathrm{span}_{\mathbb{C}}\{h_3, x_3, y_3\} \overset{\text{iso}}{=} \mathfrak{sl}(2,\mathbb{C})$

ex. 3×3 matrices with trace 0

$\boxed{\text{ΑΣΚΗΣΗ 9}}$

$$
h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
x_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
x_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad
x_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad
y_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad
y_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad
y_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$

Ideals

Definition (Ideal)

$\boxed{\textbf{Ideal}}$ $\mathcal{I}$ is a Lie subalgebra such that $\boxed{[\mathcal{I},\, \mathfrak{g}] \subset \mathcal{I}}$ $\Leftrightarrow$ $\forall x \in \mathcal{I},\ \forall y \in \mathfrak{g} \rightsquigarrow [x,y] \in \mathcal{I}$

$\boxed{\textbf{Center}}$ $\mathcal{Z}(\mathfrak{g}) \overset{\text{ορ}}{\doteq} \{z \in \mathfrak{g} \ :\ [z,\, \mathfrak{g}] = \{0\}\}$

Prop: The center is an ideal.

Prop: The **derived algebra** $\equiv \mathcal{D}\mathfrak{g} = [\mathfrak{g},\, \mathfrak{g}]$ is an ideal. $\boxed{\text{ΑΣΚΗΣΗ 10}}$

Prop: If $\mathcal{I}$, $\mathcal{J}$ are ideals $\Rightarrow \mathcal{I} + \mathcal{J}$, $[\mathcal{I},\, \mathcal{J}]$ and $\mathcal{I} \bigcap \mathcal{J}$ are ideals. $\boxed{\text{ΑΣΚΗΣΗ 11}}$

Theorem

$\mathcal{I}$ ideal of $\mathfrak{g} \rightsquigarrow \mathfrak{g}/\mathcal{I} \equiv \{\bar{x} = x + \mathcal{I} \ :\ \ x \in \mathfrak{g}\}$ is a Lie algebra

$$[\bar{x},\, \bar{y}] \overset{\text{ορ.}}{\equiv} \overline{[x,\, y]} = [x,\, y] + \mathcal{I}$$

Απόδειξη: Το $\mathfrak{g}/\mathcal{I}$ είναι γραμμικός χώρος, $[\bar{x},[\bar{y},\bar{z}]] = [\bar{x}, \overline{[y,z]}] = \overline{[x,[y,z]]} \rightsquigarrow$ $[\bar{x},[\bar{y},\bar{z}]] + \text{cycl} = \overline{[x,[y,z]] + \text{cycl}} = \bar{0} = \mathcal{I}$.

Simple Ideals

Def: $\mathfrak{g}$ is **simple** $\Leftrightarrow$ $\mathfrak{g}$ has **only** trivial ideals and $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$
(trivial ideals of $\mathfrak{g}$ are the ideals $\{0\}$ and $\mathfrak{g}$)

Def: $\mathfrak{g}$ is **abelian** $\Leftrightarrow$ $[\mathfrak{g}, \mathfrak{g}] = \{0\}$

Prop: $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian

ΑΣΚΗΣΗ 12

Prop: $\mathfrak{g}$ is **simple Lie algebra** $\Rightarrow \mathcal{Z}(\mathfrak{g}) = \{0\}$ and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

ΑΣΚΗΣΗ 13

Prop: The Classical Lie Algebras A_ℓ, B_ℓ, C_ℓ, D_ℓ are simple Lie Algebras

Θα αποδειχθεί στο τέλος του μαθήματος!

ΑΣΚΗΣΗ 14 (δύσκολη)

Να αποδειχθεί ότι η άλγεβρα $\mathfrak{sl}(2,\mathbb{C})$ είναι μια απλή Lie άλγεβρα.

Direct Sum of Lie Algebras

$\mathfrak{g}_1$, $\mathfrak{g}_1$ Lie algebras
direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{g}_1 \times \mathfrak{g}_2$ with the following structure:

$$\alpha(x_1, x_2) + \beta(y_1, y_2) \equiv (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$

$\rightsquigarrow$ $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ vector space

commutator definition: $[(x_1, x_2)\,,\,(y_1, y_2)] \equiv ([x_1,\, y_1]\,,\,[x_2,\, y_2])$

Prop: $\boxed{\mathfrak{g}_1 \oplus \mathfrak{g}_2 \text{ is a Lie algebra}}$

$$\left.\begin{array}{l} (\mathfrak{g}_1, 0) \underset{\text{iso}}{\simeq} \mathfrak{g}_1 \\ (0, \mathfrak{g}_2) \underset{\text{iso}}{\simeq} \mathfrak{g}_2 \end{array}\right\} \rightsquigarrow \left\{\begin{array}{lcl} (\mathfrak{g}_1, 0) \bigcap (0, \mathfrak{g}_2) = \{(0,0)\} & \leftrightsquigarrow & \mathfrak{g}_1 \bigcap \mathfrak{g}_2 = 0 \\ [(\mathfrak{g}_1, 0)\,,\,(0, \mathfrak{g}_2)] = \{(0,0)\} & \leftrightsquigarrow & [\mathfrak{g}_1,\, \mathfrak{g}_2] = 0 \end{array}\right.$$

Prop: $\boxed{\mathfrak{g}_1 \text{ and } \mathfrak{g}_2 \text{ are ideals of } \mathfrak{g}_1 \oplus \mathfrak{g}_2}$

Prop: $$\left\{\begin{array}{c} \mathfrak{a},\ \mathfrak{b} \text{ ideals of } \mathfrak{g} \\ \mathfrak{a} \bigcap \mathfrak{b} = \{0\} \\ \mathfrak{a} + \mathfrak{b} = \mathfrak{g} \end{array}\right\} \rightsquigarrow \left\{\mathfrak{a} \oplus \mathfrak{b} \underset{\text{iso}}{\longleftrightarrow} \mathfrak{a} + \mathfrak{b}\right\} \rightsquigarrow \left\{\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{g}\right\}$$

Lie homomorphisms

homomorphism:

$$\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}' \quad \begin{cases} \text{(i)} & \phi \text{ linear} \\ \text{(ii)} & \phi([x,y]) = [\phi(x), \phi(y)] \end{cases}$$

monomorphism: $\mathrm{Ker}\,\phi = \{0\}$, **epimorphism**: $\mathrm{Im}\,\phi = \mathfrak{g}'$

isomorphism: mono+ epi, **automorphism**: iso+ $\{\mathfrak{g} = \mathfrak{g}'\}$

Prop: $\mathrm{Ker}\,\phi$ is an ideal of $\mathfrak{g}$

Αποδ. $\phi([\mathrm{Ker}\,\phi, \mathfrak{g}]) = [\underbrace{\phi(\mathrm{Ker}\,\phi)}_{=\{0\}}, \phi(\mathfrak{g})] = \{0\} \rightsquigarrow [\mathrm{Ker}\,\phi, \mathfrak{g}] \subseteq \mathrm{Ker}\,\phi$

Prop: $\mathrm{Im}\,\phi = \phi(\mathfrak{g})$ is Lie subalgebra of $\mathfrak{g}'$

$[\phi(\mathfrak{g}), \phi(\mathfrak{g})] = \phi([\mathfrak{g}, \mathfrak{g}]) \subseteq \phi(\mathfrak{g})$

Theorem

$$\left\{ \begin{array}{l} \mathfrak{I} \textit{ ideal of } \mathfrak{g} \\ \textit{canonical map } \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{I} \\ \pi(x) = \bar{x} = x + \mathfrak{I} \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \textit{canonical map is a} \\ \textit{Lie epimorphism} \end{array} \right\}$$

Linear homomorphism theorem

Theorem

ΣΗΜΕΙΩΣΗ 7

V, U, W linear spaces
$f : V \longrightarrow U$ *and* $g : V \longrightarrow W$ *linear maps*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g"'] & U \arrow[dl, dashed, "\psi"] \\ W & \end{tikzcd} \qquad \left\{ \begin{array}{c} f \ \ \textit{epi} \\ Ker f \subset Ker g \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \exists ! \, \psi \ : \\ g = \psi \circ f \end{array} \right\}$$

$$Ker \psi = f\,(Ker g)$$

Corollary

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g"'] & U \arrow[dl, dashed, "\psi"] \\ W & \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "f"] \arrow[d, "g"'] & U \\ W \arrow[ur, dashed, "\psi^{-1}"'] & \end{tikzcd}$$

$$\left\{ \ f \ \ \textit{epi}, \quad g \ \ \textit{epi}, \quad Ker f = Ker g \ \right\} \rightsquigarrow \left\{ \ \exists ! \, \psi \ \ \textit{iso} \ : \ g = \psi \circ f, \quad U \underset{\text{iso}}{\simeq} W \ \right\}$$

$\boxed{\text{ΣΗΜΕΙΩΣΗ 7}}$

$$\begin{tikzcd} U \arrow[r, "f", "\text{epi}"'] \arrow[d, "g"'] & V \arrow[dl, "\exists!\psi"] \\ W & \end{tikzcd}$$

$\boxed{\begin{array}{l} f \text{ epi} \\ \operatorname{Ker} f \subseteq \operatorname{Ker} g \end{array}} \Rightarrow \boxed{\begin{array}{l} \exists!\psi \\ g = \psi \circ f \end{array}}$

Απόδειξη Έστω $v \in V$, f epi $\rightsquigarrow f^{-1}(v) \neq \emptyset$. Αν $u \in f^{-1}(v)$ ορίζουμε $\psi(v) = g(u)$.

Αν u_1 και u_2 ανήκουν το $f^{-1}(v) \rightsquigarrow f(u_1 - u_2) = f(u_1) - f(u_2) = v - v = 0 \rightsquigarrow$ $u_1 - u_2 \in \operatorname{Ker} f \subseteq \operatorname{Ker} g \rightsquigarrow g(u_1 - u_2) = 0 \rightsquigarrow g(u_1) = g(u_2)$

επομένως $g(f^{-1}(v)) = \psi(v)$ δ/δ υπάρχει ένα ψ τέτοιο ώστε

$\psi = g \circ f^{-1} \rightsquigarrow g = \psi \circ f$.

Lie homomorphism theorem

Theorem

$\mathfrak{g}, \mathfrak{h}, \mathfrak{l}$ *Lie spaces,* $\phi : \mathfrak{g} \underset{\text{epi}}{\longrightarrow} \mathfrak{h}$ *and* $\rho : \mathfrak{g} \longrightarrow \mathfrak{l}$ *Lie homomorphisms*

$\boxed{\text{ΣΗΜΕΙΩΣΗ 8}}$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\rho"'] & \mathfrak{h} \arrow[dl, dashed, "\psi"] \\ \mathfrak{l} & \end{tikzcd} \quad \left\{ \begin{array}{c} \phi \ \textit{Lie-epi}, \\ Ker\phi \subset Ker\rho \end{array} \right\} \rightsquigarrow \left\{ \begin{array}{c} \exists!\,\psi \ : \ \textit{Lie-homo} \\ \rho = \psi \circ \phi \end{array} \right\}$$

Corollary

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\rho"'] & \mathfrak{h} \arrow[dl, dashed, "\psi"] \\ \mathfrak{l} & \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{g} \arrow[r, "\phi"] \arrow[d, "\rho"'] & \mathfrak{h} \\ \mathfrak{l} \arrow[ur, dashed, "\psi^{-1}"'] & \end{tikzcd}$$

$$\left\{ \ \phi \ \textit{Lie-epi}, \ \rho \ \textit{Lie-epi}, \quad Ker\phi = Ker\rho \ \right\} \rightsquigarrow \left\{ \ \exists!\,\psi \ : \ \textit{Lie-iso} \ \rho = \psi \circ \phi, \ \mathfrak{h} \underset{\text{iso}}{\simeq} \mathfrak{l} \ \right\}$$

$\boxed{\text{ΣΗΜΕΙΩΣΗ 8}}$

$\mathfrak{g}, \mathfrak{h}, \mathfrak{l}$ άλγεβρες Lie

φ, ρ, ψ Lie-homomorphisms

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi", "epi"'] \arrow[d, "\rho"'] & \mathfrak{h} \arrow[dl, "\exists!\psi"] \\ \mathfrak{l} & \end{tikzcd}$$

$$\boxed{\begin{array}{c} \varphi \text{ epi} \\ Ker\varphi \subseteq Ker\rho \end{array}} \Rightarrow \boxed{\begin{array}{c} \exists!\psi \\ \rho = \psi \circ \varphi \end{array}}$$

Απόδειξη: Τα $\mathfrak{g}, \mathfrak{h}, \mathfrak{l}$ είναι γραμμικοί χώροι και τα φ, ρ, ψ γραμμικές απεικονίσεις επομένως υπάρχει μια μοναδική γραμμική απεικόνιση ψ θα αποδείξουμε ότι ψ είναι Lie-hom

δηλ. αν $h \in \mathfrak{h} \rightsquigarrow \psi([h_1, h_2]) = [\psi(h_1), \psi(h_2)]$. Έστω $h_1 = \varphi(g_1)$, $h_2 = \varphi(g_2) \rightsquigarrow$

$$\psi([h_1, h_2]) = \psi([\varphi(g_1), \varphi(g_2)]) \overset{\text{επειδή } \varphi \text{ είναι Lie-hom}}{=} \psi(\varphi([g_1, g_2])) = \psi \circ \varphi([g_1, g_2]) = \rho([g_1, g_2])$$

$$[\psi(h_1), \psi(h_2)] = [\psi \circ \varphi(g_1), \psi \circ \varphi(g_2)] = [\rho(g_1), \rho(g_2)]$$

Το ρ - Lie hom δηλ $\rho([g_1, g_2]) = [\rho(g_1), \rho(g_2)] \rightsquigarrow$

$$\psi([h_1, h_2]) = [\psi(h_1), \psi(h_2)] \Longrightarrow \psi \text{ είναι Lie-hom.}$$

First isomorphism theorem

Theorem (First isomorphism theorem)

$\phi \; : \; \mathfrak{g} \longrightarrow \mathfrak{g}'$ *Lie-homomorphism,*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\pi", "\text{epi}"'] \arrow[d, "\phi"'] & \mathfrak{g}/\operatorname{Ker}\phi \arrow[dl, dashed, "\phi'"] \\ \phi(\mathfrak{g}) & \end{tikzcd} \quad \Longrightarrow \quad \exists!\, \phi' \; : \; \mathfrak{g}/\operatorname{Ker}\phi \longrightarrow \phi(\mathfrak{g}) \subset \mathfrak{g}' \quad \textit{Lie-iso}$$

$$\phi(\mathfrak{g}) \underset{\text{iso}}{\simeq} \mathfrak{g}/\operatorname{Ker}\phi$$

Άμεση εφαρμογή του προηγούμενου.

Γνωστή πρόταση για γραμμικούς χώρους

$\boxed{\text{Εδώ αφορά Lie-άλγεβρες}}$

Noether Theorems

Theorem

$$\left.\begin{array}{c} \mathfrak{K},\ \mathfrak{L} \\ \textit{ideals of } \mathfrak{g} \\ \mathfrak{K} \subset \mathfrak{L} \end{array}\right\} \rightsquigarrow \left\{\begin{array}{ll} (i) & \mathfrak{L}/\mathfrak{K} \ \textit{ ideal of } \ \mathfrak{g}/\mathfrak{K} \\ & \\ (ii) & (\mathfrak{g}/\mathfrak{K})\,/\,(\mathfrak{L}/\mathfrak{K}) \underset{\text{iso}}{\simeq} (\mathfrak{g}/\mathfrak{L}) \end{array}\right.$$

ΣΗΜΕΙΩΣΗ 9

ΣΗΜΕΙΩΣΗ 10

Theorem (Noether Theorem (2nd isomorphism theor.))

$$\left\{\begin{array}{c} \mathfrak{K},\ \mathfrak{L} \\ \textit{ideals of } \mathfrak{g} \end{array}\right\} \rightsquigarrow \left\{(\mathfrak{K}+\mathfrak{L})\,/\mathfrak{L} \underset{\text{iso}}{\simeq} \mathfrak{K}/\,(\mathfrak{K}\cap\mathfrak{L})\right\}$$

ΣΗΜΕΙΩΣΗ 11

Parallelogram Low:

$$\begin{tikzcd} & \mathfrak{K}+\mathfrak{L} \arrow[dl, dash, dashed] \arrow[dr, equal, dashed] & \\ \mathfrak{K} \arrow[dr, equal, dashed] & & \mathfrak{L} \arrow[dl, dash, dashed] \\ & \mathfrak{K}\cap\mathfrak{L} & \end{tikzcd}$$

ΣΗΜΕΙΩΣΗ 9

$\mathfrak{g}$ Lie algebra
$\mathfrak{k}$, $\mathfrak{l}$ ideals
$\mathfrak{k} \subset \mathfrak{l}$

$\Rightarrow$

$\mathfrak{l}/\mathfrak{k}$ ideal of $\mathfrak{g}/\mathfrak{k}$.

Απόδειξη: Η απεικόνιση $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{k}$ είναι Lie-hom. Το $\mathfrak{l}$ είναι ιδεώδες του $\mathfrak{g}$ επομένως

$$[\mathfrak{l}, \mathfrak{g}] \subseteq \mathfrak{l} \rightsquigarrow \pi[\mathfrak{l}, \mathfrak{g}] = [\pi(\mathfrak{l}), \pi(\mathfrak{g})] \subseteq \pi(\mathfrak{l})$$

$\pi(\mathfrak{l}) = \mathfrak{l}/\mathfrak{k}$, $\pi(\mathfrak{g}) = \mathfrak{g}/\mathfrak{k}$ επομένως

$$[\mathfrak{l}/\mathfrak{k}, \mathfrak{g}/\mathfrak{k}] \subseteq \mathfrak{l}/\mathfrak{k}.$$

ΣΗΜΕΙΩΣΗ 10

$$\boxed{\begin{array}{c}\mathfrak{g} \text{ Lie algebra} \\ \mathbb{K}, \mathbb{L} \text{ ideals} \\ \mathbb{K} \subset \mathbb{L}\end{array}} \Rightarrow \boxed{(\mathfrak{g}/\mathbb{K})/(\mathbb{L}/\mathbb{K}) \underset{iso}{\cong} (\mathfrak{g}/\mathbb{L})}$$

Απόδειξη

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\pi", "epi"'] \arrow[d, "\sigma"', "epi"] & \mathfrak{g}/\mathbb{K} \arrow[r, "\rho", "epi"'] & (\mathfrak{g}/\mathbb{K})/(\mathbb{L}/\mathbb{K}) \\
\mathfrak{g}/\mathbb{L} & &
\end{tikzcd}$$

ή

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\rho\circ\pi", "epi"'] \arrow[d, "\sigma"', "epi"] & (\mathfrak{g}/\mathbb{K})/(\mathbb{L}/\mathbb{K}) \arrow[dl, "\exists!\psi"] \\
\mathfrak{g}/\mathbb{L} &
\end{tikzcd}$$

Εχουμε αν $\boxed{x \in \ker \rho\circ\pi} \rightsquigarrow \rho(\pi(x)) = 0 \rightsquigarrow \pi(x) \in \ker \rho$

$\ker \rho = \mathbb{L}/\mathbb{K} \rightsquigarrow \pi(x) \in \pi(\mathbb{L}) \rightsquigarrow x \in \mathbb{L} \rightsquigarrow \boxed{\ker(\rho\circ\pi) \subseteq \mathbb{L}}$

αν $x \in \mathbb{L} \rightsquigarrow \pi(x) \in \mathbb{L}/\mathbb{K} = \pi(\mathbb{L}) \rightsquigarrow \rho(\pi(x)) = 0 \rightsquigarrow \boxed{\mathbb{L} \subseteq \ker(\rho\circ\pi)}$ $\Rightarrow \boxed{\mathbb{L} = \ker(\rho\circ\pi)}$

Επίσης $\mathbb{L} = \ker \sigma$. Επομένως ψ isomorphism $\rightsquigarrow$ $\boxed{(\mathfrak{g}/\mathbb{K})/(\mathbb{L}/\mathbb{K}) \overset{iso}{=} \mathfrak{g}/\mathbb{L}}$

ΣΗΜΕΙΩΣΗ 11

$\mathbb{K}, \mathbb{L}$ ideals of $\mathfrak{g}$ $\Rightarrow$ $(\mathbb{K}+\mathbb{L})/\mathbb{L} \overset{iso}{\cong} \mathbb{K}/\mathbb{K}\cap\mathbb{L}$

Απόδειξη: $(\mathbb{K}+\mathbb{L})/\mathbb{K}\cap\mathbb{L} = (\mathbb{K}/\mathbb{K}\cap\mathbb{L}) \oplus (\mathbb{L}/\mathbb{K}\cap\mathbb{L})$ δ.δ. $(\mathbb{K}/\mathbb{K}\cap\mathbb{L})\cap(\mathbb{L}/\mathbb{K}\cap\mathbb{L}) = \{\bar{0}\}$

Έστω $\mathbb{K}+\mathbb{L} \xrightarrow{\pi} (\mathbb{K}+\mathbb{L})/\mathbb{K}\cap\mathbb{L}$ και $\boxed{\bar{x} \neq \bar{0}}$, όπου $\bar{0} \overset{def}{\equiv} \mathbb{K}\cap\mathbb{L}$ και $\boxed{\bar{x} \in (\mathbb{K}/\mathbb{K}\cap\mathbb{L})\cap(\mathbb{L}/\mathbb{K}\cap\mathbb{L}) \neq \{\bar{0}\}}$ ← Υπόθεση

και $\bar{x} = \pi(x)$

Τότε $\bar{x} = \pi(x) \in \mathbb{K}/\mathbb{K}\cap\mathbb{L} \rightsquigarrow x \in \mathbb{K}$ και $\bar{x} = \pi(x) \in \mathbb{L}/\mathbb{K}\cap\mathbb{L} \rightsquigarrow x \in \mathbb{L}$ οπότε $x \in \mathbb{K}\cap\mathbb{L}$

$\rightsquigarrow$ $\boxed{\bar{x} = \pi(x) = \{0\}}$ ← Συμπέρασμα "Άτοπο"

Επομένως

$$((\mathbb{K}+\mathbb{L})/\mathbb{K}\cap\mathbb{L})/(\mathbb{K}/\mathbb{K}\cap\mathbb{L}) = \mathbb{L}/\mathbb{K}\cap\mathbb{L}$$ και από την προηγούμε πρόταση

$$((\mathbb{K}+\mathbb{L})/\mathbb{K}\cap\mathbb{L})/(\mathbb{K}/\mathbb{K}\cap\mathbb{L}) \overset{iso}{\cong} (\mathbb{K}+\mathbb{L})/\mathbb{K}$$

άρα $(\mathbb{K}+\mathbb{L})/\mathbb{K} \overset{iso}{\cong} \mathbb{L}/\mathbb{K}\cap\mathbb{L}$